



5.7 Projections and best approximations





Introduction

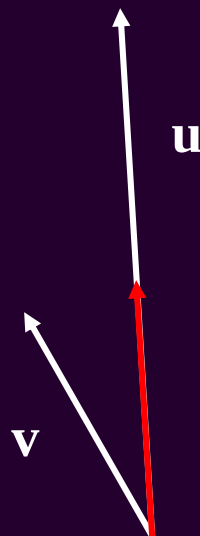
- In this topic, we will
 - Define the projection of one vector onto another
 - This will be the best approximation of one vector as a scalar multiple of another
 - First find the formula for real finite-dimensional vectors and then discuss the formula for complex finite-dimensional vectors
 - Look at some examples and properties
 - Look at projections of functions onto functions
 - Discuss the average value of vectors





Projections for real inner-product spaces

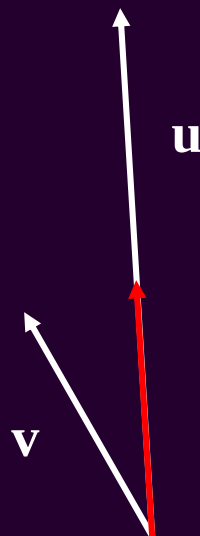
- Now, given two vectors $\mathbf{u}$ and $\mathbf{v}$,
clearly $\mathbf{u}$ is not a very good approximation of $\mathbf{v}$
 - However, $0.4\mathbf{u}$ seems to be a better approximation of $\mathbf{v}$





Projections for real inner-product spaces

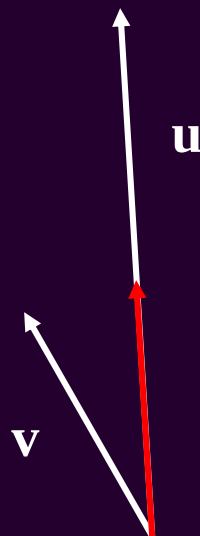
- Question: Which scalar multiple of $\mathbf{u}$ is the best approximation of $\mathbf{v}$?
 - For this, we need a definition of “best”





Projections for real inner-product spaces

- Most reasonable: Which scalar multiple of $\mathbf{u}$ is closest to $\mathbf{v}$
 - That is, find that multiple that minimizes $\|\alpha\mathbf{u} - \mathbf{v}\|_2$



- Recall that the function x^2 is monotonic for $x \geq 0$,
that is, $x < y$ if and only if $x^2 < y^2$





Projections for real inner-product spaces

- Thus, finding a minimum of $\|\alpha \mathbf{u} - \mathbf{v}\|_2$ is equivalent to finding a minimum of

$$\begin{aligned}\|\alpha \mathbf{u} - \mathbf{v}\|_2^2 &= \langle \alpha \mathbf{u} - \mathbf{v}, \alpha \mathbf{u} - \mathbf{v} \rangle \\ &= \alpha \langle \mathbf{u}, \alpha \mathbf{u} - \mathbf{v} \rangle - \langle \mathbf{v}, \alpha \mathbf{u} - \mathbf{v} \rangle \\ &= \alpha (\alpha \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle) - (\alpha \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle) \\ &= \alpha^2 \langle \mathbf{u}, \mathbf{u} \rangle - \alpha \langle \mathbf{u}, \mathbf{v} \rangle - \alpha \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= \alpha^2 \langle \mathbf{u}, \mathbf{u} \rangle - 2\alpha \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle\end{aligned}$$

$$\langle \alpha \mathbf{u} - \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle$$





Projections for real inner-product spaces

- Therefore,

$$\|\alpha \mathbf{u} - \mathbf{v}\|_2^2 = \alpha^2 \langle \mathbf{u}, \mathbf{u} \rangle - 2\alpha \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle$$

- Remember that $\mathbf{u}$ and $\mathbf{v}$ are given, so

$$\langle \mathbf{u}, \mathbf{u} \rangle, -2\langle \mathbf{u}, \mathbf{v} \rangle \text{ and } \langle \mathbf{v}, \mathbf{v} \rangle$$

are all constants!

- For what value of α is $c_2\alpha^2 + c_1\alpha + c_0$ a minimum?

- This assumes $c_2 > 0$

- Differentiate, equate to zero and solve for α :

$$2c_2\alpha + c_1 = 0$$

- Thus, $\alpha = -\frac{c_1}{2c_2}$





Projections for real inner-product spaces

- Substituting in the two coefficients,

$$\begin{aligned}\alpha &= -\frac{c_1}{2c_2} & \|\alpha \mathbf{u} - \mathbf{v}\|_2^2 &= \alpha^2 \langle \mathbf{u}, \mathbf{u} \rangle - 2\alpha \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\ &= -\frac{-2\langle \mathbf{u}, \mathbf{v} \rangle}{2\langle \mathbf{u}, \mathbf{u} \rangle} \\ &= \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\end{aligned}$$

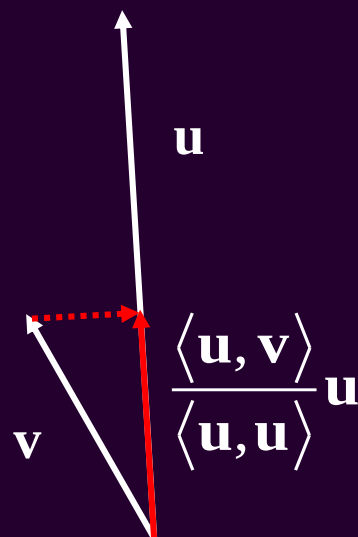




Projections for real inner-product spaces

- Thus, the best approximation of $\mathbf{v}$ as a scalar multiple of $\mathbf{u}$ is

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$





Projections for real inner-product spaces

- Note that this formula assumes that $\mathbf{u} \neq \mathbf{0}$.
 - Let us define

$$\text{proj}_{\mathbf{0}}(\mathbf{v}) = \mathbf{0}$$

- That is, the best approximation of any vector by a multiple of the zero vector is the zero vector...

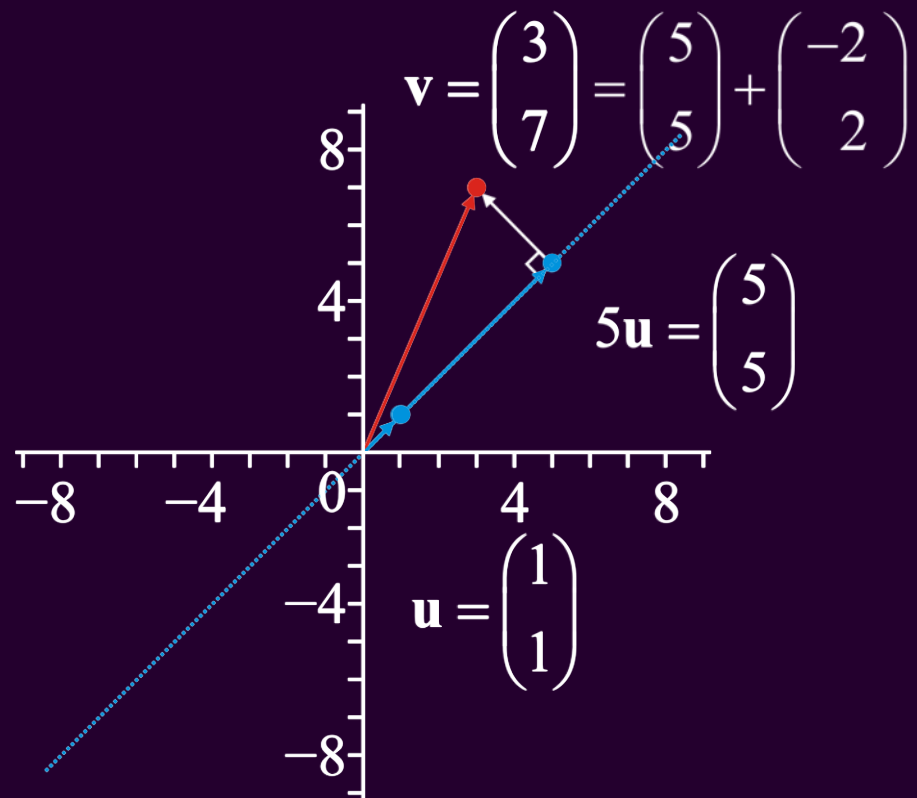




Projections for real inner-product spaces

- For example, given $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$$\begin{aligned}\text{proj}_{\mathbf{u}}(\mathbf{v}) &= \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ &= \frac{1 \cdot 3 + 1 \cdot 7}{1^2 + 1^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \frac{10}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}\end{aligned}$$





Projections for real inner-product spaces

- For example, given $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

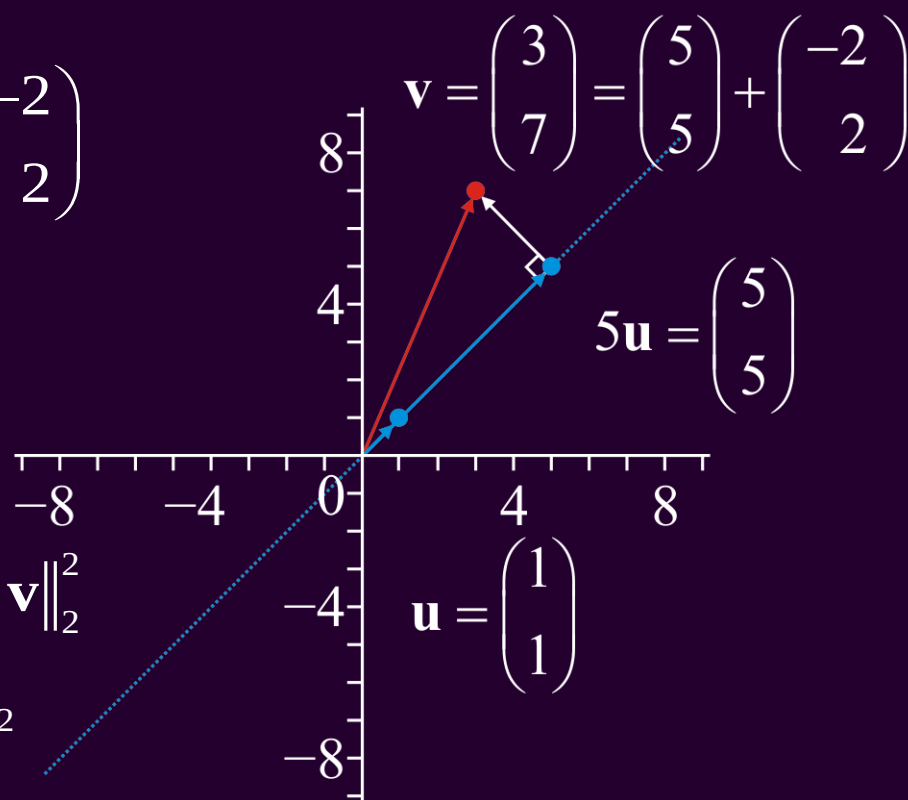
$$\text{perp}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) \perp \text{perp}_{\mathbf{u}}(\mathbf{v})$$

$$\mathbf{v} = \text{proj}_{\mathbf{u}}(\mathbf{v}) + \text{perp}_{\mathbf{u}}(\mathbf{v})$$

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2 + \|\text{perp}_{\mathbf{u}}(\mathbf{v})\|_2^2 = \|\mathbf{v}\|_2^2$$

$$5^2 + 5^2 + 2^2 + 2^2 = 3^2 + 7^2$$





Projections for real inner-product spaces

- For example, given $\mathbf{u} = \begin{pmatrix} 2.0 \\ -0.9 \\ -1.2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

$$= \frac{2 \cdot (-4) + (-0.9) \cdot 2 + (-1.2) \cdot 2}{2^2 + 0.9^2 + 1.2^2} \begin{pmatrix} 2.0 \\ -0.9 \\ -1.2 \end{pmatrix}$$

$$= \frac{-1.952}{6.25} \begin{pmatrix} 2.0 \\ -0.9 \\ -1.2 \end{pmatrix} = \begin{pmatrix} -3.904 \\ 1.7568 \\ 2.3424 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} -3.904 \\ 1.7568 \\ 2.3424 \end{pmatrix} + \begin{pmatrix} -0.096 \\ 0.2432 \\ -0.3424 \end{pmatrix}$$

$$2(-0.096) + (-0.9)(0.2432) + (-1.2)(-0.3424) = 0$$





Projections for real inner-product spaces

- For example, given $\mathbf{u} = \begin{pmatrix} 0.6 \\ -1.8 \\ 1.2 \\ -1.1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -5 \\ 4 \\ -4 \end{pmatrix}$

$$\begin{aligned} \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ &= \frac{0.6 \cdot 3 + (-1.8) \cdot (-4) + 1.2 \cdot 4 + (-1.1) \cdot (-4)}{0.6^2 + 1.8^2 + 1.2^2 + 1.1^2} \begin{pmatrix} 0.6 \\ -1.8 \\ 1.2 \\ -1.1 \end{pmatrix} \\ &= \frac{3.2}{6.25} \begin{pmatrix} 0.6 \\ -1.8 \\ 1.2 \\ -1.1 \end{pmatrix} = \begin{pmatrix} 1.92 \\ -5.76 \\ 3.84 \\ -3.52 \end{pmatrix} \\ \mathbf{v} &= \begin{pmatrix} 1.92 \\ -5.76 \\ 3.84 \\ -3.52 \end{pmatrix} + \begin{pmatrix} 1.08 \\ 0.76 \\ 0.16 \\ -0.48 \end{pmatrix} \end{aligned}$$





Projections for complex inner-product spaces

- In finding the projection for real inner-product spaces, we used the symmetry property:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

- Complex vectors, however, do not have this property:
they are *conjugate symmetric*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle^*$$

- Fortunately, the same formula holds, and the detailed derivation is in the text book:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$





Properties of projections

- A function f is *idempotent* if $f(f(x)) = f(x)$ for all x
 - That is, once the function is applied, subsequent applications do not affect the result
 - You already know many idempotent function:
 - The absolute value function $\|x\| = |x|$
 - The ceiling function $\lceil \lceil x \rceil \rceil = \lceil x \rceil$
 - The floor function $\lfloor \lfloor x \rfloor \rfloor = \lfloor x \rfloor$





Properties of projections

Theorem:

The projection is idempotent; that is,

$$\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}(\mathbf{v})) = \text{proj}_{\mathbf{u}}(\mathbf{v})$$

Proof:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

$$\text{proj}_{\mathbf{u}}\left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}\right) = \frac{\left\langle \mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \right\rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

But $\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}$ is a scalar

and $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$

$$= \frac{\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \langle \mathbf{u}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} = \text{proj}_{\mathbf{u}}(\mathbf{v}) \blacksquare$$





Properties of projections

- A function f is *linear* if $f(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v})$
for all possible values of $\alpha, \beta, \mathbf{u}$ and $\mathbf{v}$
 - Consider the function
$$f(\mathbf{u}) \stackrel{\text{def}}{=} 3\mathbf{u}$$
 - Note that
$$\begin{aligned} f(2\mathbf{u} + 5\mathbf{v}) &= 3(2\mathbf{u} + 5\mathbf{v}) \\ &= 3(2\mathbf{u}) + 3(5\mathbf{v}) \\ &= 2(3\mathbf{u}) + 5(3\mathbf{v}) \\ &= 2f(\mathbf{u}) + 5f(\mathbf{v}) \end{aligned}$$





Properties of projections

- More useful, the derivative is linear:

$$\begin{aligned}\frac{d}{dx}(\alpha f(x) + \beta g(x)) &= \frac{d}{dx}(\alpha f(x)) + \frac{d}{dx}(\beta g(x)) \\ &= \alpha \frac{d}{dx} f(x) + \beta \frac{d}{dx} g(x)\end{aligned}$$

- For example,

$$\begin{aligned}\frac{d}{dx}(3\cos(x) + 5\sin(x)) &= \frac{d}{dx}(3\cos(x)) + \frac{d}{dx}(5\sin(x)) \\ &= 3\frac{d}{dx}\cos(x) + 5\frac{d}{dx}\sin(x) \\ &= -3\sin(x) + 5\cos(x)\end{aligned}$$





Properties of projections

Theorem:

The projection is linear.

Proof:

$$\begin{aligned}\text{proj}_{\mathbf{u}}(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) &= \frac{\langle \mathbf{u}, \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ &= \frac{\alpha_1 \langle \mathbf{u}, \mathbf{v}_1 \rangle + \alpha_2 \langle \mathbf{u}, \mathbf{v}_2 \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ &= \alpha_1 \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} + \alpha_2 \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} \\ &= \alpha_1 \text{proj}_{\mathbf{u}}(\mathbf{v}_1) + \alpha_2 \text{proj}_{\mathbf{u}}(\mathbf{v}_2) \blacksquare\end{aligned}$$





Properties of projections

Theorem:

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2 \leq \|\mathbf{v}\|_2 \text{ and}$$

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2 = \|\mathbf{v}\|_2 \text{ if and only if } \mathbf{v} = \alpha \mathbf{u} \text{ i.e., } \text{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v}$$

Proof:

$$\text{First, } \text{proj}_{\mathbf{u}}(\mathbf{v}) + \text{perp}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v}$$

But, $\text{proj}_{\mathbf{u}}(\mathbf{v}) \perp \text{perp}_{\mathbf{u}}(\mathbf{v})$ so apply the Pythagorean theorem

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2 + \|\text{perp}_{\mathbf{u}}(\mathbf{v})\|_2^2 = \|\mathbf{v}\|_2^2$$

Because each of these terms is non-negative,

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2 \leq \|\mathbf{v}\|_2^2$$

Because the square-root function is monotonic for $x \geq 0$,

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2 \leq \|\mathbf{v}\|_2$$





Properties of projections

Next, if $\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2 = \|\mathbf{v}\|_2$, then $\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|_2^2 = \|\mathbf{v}\|_2^2$

By the Pythagorean theorem, $\|\text{perp}_{\mathbf{u}}(\mathbf{v})\|_2 = 0$ so

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) + \mathbf{0} = \mathbf{v}$$

and the projection is a scalar multiple of $\mathbf{u}$.

Otherwise, assume $\mathbf{v} = \alpha \mathbf{u}$, so

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \text{proj}_{\mathbf{u}}(\alpha \mathbf{u}) = \alpha \text{proj}_{\mathbf{u}}(\mathbf{u}) = \alpha \frac{\langle \mathbf{u}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} = \alpha \mathbf{u} = \mathbf{v} \quad \blacksquare$$





Properties of projections

Theorem:

If $\hat{\mathbf{u}}$ is a unit vector, then $\text{proj}_{\hat{\mathbf{u}}}(\mathbf{v}) = \langle \hat{\mathbf{u}}, \mathbf{v} \rangle \hat{\mathbf{u}}$.

Proof:

$$\text{proj}_{\hat{\mathbf{u}}}(\mathbf{v}) = \frac{\langle \hat{\mathbf{u}}, \mathbf{v} \rangle}{\langle \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle} \hat{\mathbf{u}}$$

But $\langle \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle = \|\hat{\mathbf{u}}\|_2^2$, and if $\hat{\mathbf{u}}$ is a unit vector, $\|\hat{\mathbf{u}}\|_2 = 1$,
Therefore, $\|\hat{\mathbf{u}}\|_2^2 = \langle \hat{\mathbf{u}}, \hat{\mathbf{u}} \rangle = 1$, so

$$\text{proj}_{\hat{\mathbf{u}}}(\mathbf{v}) = \frac{\langle \hat{\mathbf{u}}, \mathbf{v} \rangle}{1} \hat{\mathbf{u}} = \langle \hat{\mathbf{u}}, \mathbf{v} \rangle \hat{\mathbf{u}} \quad \blacksquare$$





Properties of projections

Theorem:

$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{0}$ if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Proof:

If $\mathbf{u} = \mathbf{0}$, then this is obviously true, and the zero vector is perpendicular to all vectors.

$$\begin{aligned} \text{If } \mathbf{u} \neq \mathbf{0}, \text{proj}_{\mathbf{u}}(\mathbf{v}) = \mathbf{0} &\Leftrightarrow \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u} = \mathbf{0} \\ &\Leftrightarrow \langle \mathbf{u}, \mathbf{v} \rangle = 0 \quad \blacksquare \end{aligned}$$

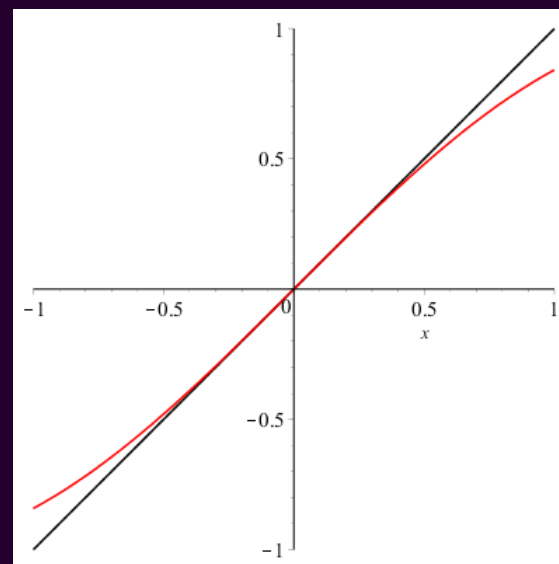




Projections in other vector spaces

- We will look at projections in one other vector space
 - You are not responsible for this!
- Let us find the best scalar multiple of x that approximates the sine function on the interval $[-1, 1]$
 - Now, $y = x$ is a great approximation of $\sin(x)$ close to $x = 0$
 - Unfortunate, the error is larger closer to the boundaries

$$\begin{aligned}\text{proj}_{\hat{x}}(\sin) &= \frac{\langle \hat{x}, \sin \rangle}{\langle \hat{x}, \hat{x} \rangle} \hat{x} \\ &= \frac{\int_{-1}^1 x \sin(x) dx}{\int_{-1}^1 x^2 dx} \hat{x}\end{aligned}$$



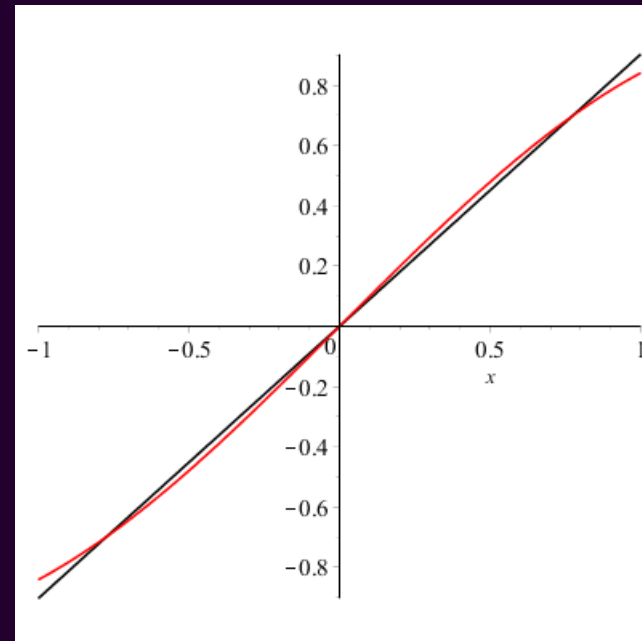
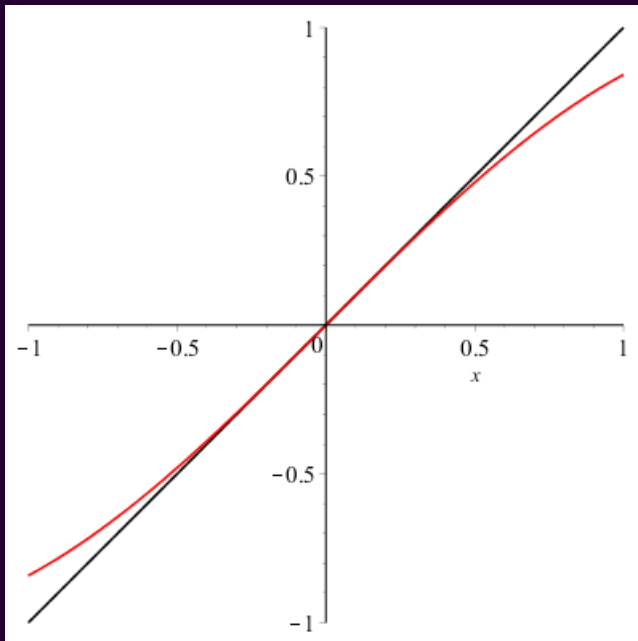


Projections in other vector spaces

- Evaluating these, we get:

$$\text{proj}_{\hat{x}}(\sin) = \frac{2\sin(1) - 2\cos(1)}{\frac{2}{3}} \hat{x}$$

- This is approximately the polynomial $0.9035x$





Projections in other vector spaces

- Let us find the constant function that approximates the sine function on the interval $[-1, 1]$
 - Every constant function is a multiple of $y = 1$, that is, $\hat{1}$

$$\begin{aligned}\text{proj}_{\hat{1}}(\sin) &= \frac{\langle \hat{1}, \sin \rangle}{\langle \hat{1}, \hat{1} \rangle} \hat{1} \\ &= \frac{\int_{-1}^1 1 \cdot \sin(x) dx}{\int_{-1}^1 1^2 dx} \hat{1} = \hat{0}\end{aligned}$$





Best approximation by constant vector

- Note that if $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, then the best approximation of any vector by a scalar multiple of $\mathbf{1}$ is the average value of the entries

$$\langle \mathbf{1}, \mathbf{v} \rangle = \sum_{k=1}^n (1 \cdot v_k) = \sum_{k=1}^n v_k \quad \langle \mathbf{1}, \mathbf{1} \rangle = \sum_{k=1}^n (1 \cdot 1) = n$$

$$\text{-- Therefore, } \text{proj}_{\mathbf{1}}(\mathbf{v}) = \frac{\langle \mathbf{1}, \mathbf{v} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} \mathbf{1} = \frac{\sum_{k=1}^n v_k}{n} \mathbf{1}$$





Summary

- Following this topic, you now
 - Know the formula for the projection of one vector onto another
 - Understand that the projection is the best approximation of one vector by a scalar multiple of another
 - Know that
 - The projection is idempotent and linear
 - The 2-norm of the projection of $\mathbf{v}$ onto another vector is always less than the 2-norm of $\mathbf{v}$
 - The formula of the projection onto $\mathbf{u}$ simplifies if $\mathbf{u}$ is a unit vector
 - The projection of $\mathbf{v}$ onto $\mathbf{u}$ is $\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular
 - Have seen projections of functions and other results





References

- [1] [https://en.wikipedia.org/wiki/Projection_\(linear_algebra\)](https://en.wikipedia.org/wiki/Projection_(linear_algebra))
- [2] <https://en.wikipedia.org/wiki/Orthogonality>





Acknowledgments

Edward Hong for noting I forgot to cancel an α on Page 6 after it was factored out.

Mac Qiu for noting one slide had the vector $(2, -2)$ instead of $(-2, 2)$





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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